Some equations that are scaled to the square equation

1) Type of equation $ax^4+bx^2+c=0$

This type of equation we solve with replacement $x^2=t$, get equation $at^2+bt+c=0$,and find $t_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$; then we go back to replacement:

$x^2=t_1$ and $x^2=t_2$

$x_{1,2}=\pm\sqrt{t_1}$ and $x_{3,4}=\pm\sqrt{t_2}$

Example 1. Solve the equation: $x^4-4x^2+3=0$

Solution:

$x^4-4x^2+3=0\Rightarrow$ replacement $x^2=t$

$t^2-4t+3=0$

$a=1$

$b=-4$

$c=3$

$$t_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-4)\pm\sqrt{(-4)^2-4\cdot1\cdot3}}{2\cdot1}$$

$$t_{1,2}=\frac{4\pm\sqrt{16-12}}{2}=\frac{4\pm2}{2}$$

$$t_1=\frac{4+2}{2}=3$$

$$t_2=\frac{4-2}{2}=1$$

Back in $x^2=t$:

$x^2=t_1$

$x^2=3$

$x_{1,2}=\pm\sqrt{3}$

$x_1=+\sqrt{3}$

$x_2=-\sqrt{3}$

and

$x^2=t_2$

$x^2=1$

$x_{3,4}=\pm\sqrt{1}$

$x_3=+1$

$x_4=-1$

Example 2. Solve the equation: $(4x^2-5)^2+(x^2+5)^2=2(8x^4-83)$

Solution:

$$(4x^2-5)^2+(x^2+5)^2=2(8x^4-83)$$
$$16x^4-40x^2+25+x^4+10x^2+25=16x^4-166$$
$$x^4-30x^2+50+166=0$$
$$x^4-30x^2+216=0 \rightarrow \text{replacement}: \; x^2=t$$
$$t^2+30t+216=0$$

$a=1$

$b=-30$

$c=216$

$$t_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{30\pm\sqrt{900-864}}{2}$$
$$t_{1,2}=\frac{30\pm 6}{2}$$
$$t_1=\frac{36}{2}=18$$
$$t_2=\frac{24}{2}=12$$

Now:

$x^2=18$	$x^2=12$
$x_{1,2}=\pm\sqrt{18}$	$x_{3,4}=\pm\sqrt{12}$
$x_{1,2}=\pm 3\sqrt{2}$	$x_{3,4}=\pm 2\sqrt{2}$
$x_1=+3\sqrt{2}$	$x_3=+2\sqrt{2}$
$x_2=-3\sqrt{2}$	$x_4=-2\sqrt{2}$

Example 3. Solve the equation: $(x^2-2x)^2-2(x^2-2x)=3$

Solution:

It is better to take the replacement: $x^2-2x=t$

$$(x^2-2x)^2-2(x^2-2x)=3$$
$$\boxed{(x^2-2x)}^2-2\boxed{(x^2-2x)}=3$$
$$t^2-2t=3$$
$$t^2-2t-3=0$$

$a=1$

$b=-2$ →

$c=-3$

$$t_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$t_{1,2}=\frac{2\pm\sqrt{4+12}}{2}=\frac{2\pm 4}{2}$$
$$t_1=3$$
$$t_2=-1$$

Let's go back now to $x^2 - 2x = t$

$$x^2 - 2x = t_1$$
$$x^2 - 2x = 3$$
$$x^2 - 2x - 3 = 0$$

$$x^2 - 2x = t_2$$
$$x^2 - 2x = -1$$
$$x^2 - 2x + 1 = 0$$

We have now two new square equation :

$x^2 - 2x - 3 = 0 \longrightarrow$ $a = 1$, $b = -2$, $c = -3$ $\longrightarrow$

$$x_{1,2} = \frac{2 \pm \sqrt{4+12}}{2}$$
$$x_{1,2} = \frac{2 \pm 4}{2}$$
$$x_1 = 3$$
$$x_2 = -1$$

$x^2 - 2x + 1 = 0 \longrightarrow$ $a = 1$, $b = -2$, $c = 1$ $\longrightarrow$

$$x_{3,4} = \frac{2 \pm \sqrt{4-4}}{2}$$
$$x_{3,4} = \frac{2 \pm 0}{2}$$
$$x_3 = 1$$
$$x_4 = 1$$

Therefore, solutions are: $\{3, -1, 1, 1\}$

Example 4: Solve the equation: $x(x+1)(x+2)(x+3) = 0{,}5625$

Solution: If we multiply all , then we have the problem!

Try to multiply the first two, and the other two, to see what will drop ..

$$(x^2 + x)(x^2 + 3x + 2x + 6) = 0{,}5625$$
$$(x^2 + x)(x^2 + 5x + 6) = 0{,}5625 \rightarrow \text{ Not good!}$$

Try to multiply the first and fourth, and second and third

$$x(x+1)(x+2)(x+3) = 0{,}5625$$

$$(x^2 + 3x)(x^2 + 2x + 1x + 2) = 0{,}5625$$
$$(x^2 + 3x)(x^2 + 3x + 2) = 0{,}5625$$

Well, this is the better $\Rightarrow$ replacement: $x^2+3x=t$

$$t\cdot(t+2)=0{,}5625$$
$$t^2+2t-0{,}5625=0$$
$$t_{1,2}=\frac{-2\pm\sqrt{4+2{,}25}}{2}=\frac{-2\pm 2{,}5}{2}$$
$$t_1=+0{,}25$$
$$t_2=-2{,}25$$

$$x^2+3x=t_1$$
$$x^2+3x=+0{,}25$$
$$x^2+3x-0{,}25=0$$
$$x_{1,2}=\frac{-3\pm\sqrt{9+1}}{2}$$
$$x_{1,2}=\frac{-3\pm\sqrt{10}}{2}$$
$$x_1=\frac{-3+\sqrt{10}}{2}$$
$$x_2=\frac{-3-\sqrt{10}}{2}$$

$$x^2+3x=t_2$$
$$x^2+3x=-2{,}25$$
$$x^2+3x+2{,}25=0$$
$$x_{3,4}=\frac{-3\pm\sqrt{9-9}}{2}$$
$$x_{3,4}=\frac{-3\pm\sqrt{0}}{2}$$
$$x_3=x_4=-\frac{3}{2}$$

Example 5. Solve the equation: $\dfrac{x^2+x-5}{x}+\dfrac{3x}{x^2+x-5}+4=0$

Solution:

$$\frac{x^2+x-5}{x}+\frac{3x}{x^2+x-5}+4=0$$
$$\frac{x^2+x-5}{x}+3\cdot\frac{x}{x^2+x-5}+4=0$$

It is useful to take replacement $\dfrac{x^2+x-5}{x}=t$ because then $\dfrac{x}{x^2+x-5}=\dfrac{1}{t}$

$$t+3\cdot\frac{1}{t}+4=0\,/\cdot t$$
$$t^2+3+4t=0$$
$$t^2+4t+3=0$$

$$t_{1,2}=\frac{-4\pm\sqrt{16-12}}{2}=\frac{-4\pm 2}{2}$$
$$t_1=-1$$
$$t_2=-3$$

Back in the replacement $\frac{x^2+x-5}{x}=t$

$$\frac{x^2+x-5}{x}=-1 \quad \text{or} \quad \frac{x^2+x-5}{x}=-3$$

$$x^2+x-5=-x$$
$$x^2+x-5+x=0$$
$$x^2+2x-5=0$$
$$x_{1,2}=\frac{-2\pm\sqrt{4+20}}{2}$$
$$x_{1,2}=\frac{-2\pm\sqrt{24}}{2}$$
$$x_{1,2}=\frac{-2\pm 2\sqrt{6}}{2}$$
$$x_{1,2}=\frac{2\left(-1\pm\sqrt{6}\right)}{2}$$
$$x_1=-1+\sqrt{6}$$
$$x_2=-1-\sqrt{6}$$

$$x^2+x-5=-3x$$
$$x^2+x-5+3x=0$$
$$x^2+4x-5=0$$
$$x_{3,4}=\frac{-4\pm\sqrt{16+20}}{2}$$
$$x_{3,4}=\frac{-4\pm 6}{2}$$
$$x_3=1$$
$$x_4=-5$$

$\left\{-1+\sqrt{6};-1-\sqrt{6},1,-5\right\}$ are solutions

2) Binomial equations $Ax^n \pm B=0$

These are the types of equations: $Ax^n \pm B=0$ where $A>0$ and $B>0$.

First, we try to separate the equation ,using well-known formula , in factors , and use:

$M\cdot N=0 \Leftrightarrow M=0$ v $N=0$

This equation can be always solved by the replacement $x=y\sqrt[n]{\frac{B}{A}}$, when we get shape $y^n\pm 1=0$

Example 1. Solve the equation: $8x^3-27=0$

Solution:

$$8x^3-27=0$$
$$(2x)^3-3^3=0$$

Watch out: it is wrong $8x^3 = 27$ because we " lost" solutions!

$$x^3 = \frac{27}{8}$$

$$x = \sqrt[3]{\frac{27}{8}}$$

$$x = \frac{3}{2}$$

$$8x^3 - 27 = 0$$

$$(2x)^3 - 3^3 = 0$$

Use formula:

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

$$(2x-3)((2x)^2 + 2x \cdot 3 + 3^2) = 0$$

$$(2x-3)(4x^2 + 6x + 9) = 0 \Rightarrow$$

$$2x - 3 = 0$$

$$2x = 3$$

$$x_1 = \frac{3}{2}$$

or

$$4x^2 + 6x + 9 = 0$$

$$x_{2,3} = \frac{-6 \pm \sqrt{36-144}}{8}$$

$$x_{2,3} = \frac{-6 \pm \sqrt{-108}}{8} = \frac{-6 \pm 6\sqrt{3}i}{8}$$

$$x_2 = \frac{-6 + 6\sqrt{3}i}{8}$$

$$x_3 = \frac{-6 - 6\sqrt{3}i}{8}$$

$$x_2 = \frac{-6 + 6\sqrt{3}i}{8} = \frac{2(-3 + 3\sqrt{3}i)}{8} = \frac{-3 + 3\sqrt{3}i}{4}$$

$$x_3 = \frac{-6 - 6\sqrt{3}i}{8} = \frac{2(-3 - 3\sqrt{3}i)}{8} = \frac{-3 - 3\sqrt{3}i}{4}$$

Take heed: $\sqrt{-108} = \sqrt{108} \cdot \sqrt{-1} = \sqrt{36 \cdot 3} \cdot i = 6\sqrt{3}i$

Example 2. Solve the equation: $x^6 - 729 = 0$

Solution:

$$x^6 - 729 = 0$$
$$x^6 - 3^6 = 0$$
$$(x^3)^2 - (3^3)^2 = 0 \rightarrow \text{formula} \quad A^2 - B^2 = (A-B)(A+B)$$

$$(x^3 - 3^3)(x^3 + 3^3) = 0$$
$$(x-3)(x^2+3x+9)(x+3)(x^2-3x+9) = 0$$

$$x - 3 = 0 \quad \text{or} \quad x^2 + 3x + 9 = 0 \quad \text{or} \quad x + 3 = 0 \quad \text{or} \quad x^2 - 3x + 9 = 0$$

$$\boxed{x_1 = 3} \qquad x_{2,3} = \frac{-3 \pm \sqrt{9-36}}{2}$$

$$x_{2,3} = \frac{-3 \pm \sqrt{-27}}{2} = \frac{-3 \pm 3\sqrt{3}i}{8}$$

$$\boxed{x_2 = \frac{-3+3\sqrt{3}i}{2}} \qquad \boxed{x_3 = \frac{-3-3\sqrt{3}i}{2}}$$

$$x + 3 = 0 \rightarrow \boxed{x_4 = -3} \qquad x^2 - 3x + 9 = 0 \rightarrow x_{5,6} = \frac{3 \pm \sqrt{9-36}}{2}$$

$$\boxed{x_5 = \frac{3+3\sqrt{3}i}{2}} \qquad \boxed{x_6 = \frac{3-3\sqrt{3}i}{2}}$$

Example 3. Solve the equation: $5x^3+2=0$

Solution:

replacement : $x=y\sqrt[n]{\frac{B}{A}}$ $\longrightarrow$ A=5, B=2, $n=3$

$$x=y\sqrt[3]{\frac{2}{5}}$$

$$5\cdot\left(y\sqrt[3]{\frac{2}{5}}\right)^3+2=0$$

$$5\cdot y^3\cdot\frac{2}{5}+2=0$$

$$y^3\cdot 2+2=0$$

$$2\cdot(y^3+1)=0\Rightarrow y^3+1=0$$

$$(y+1)(y^2-y-1)=0$$

$$y+1=0$$

$$y_1=-1$$

or

$$y^2-y-1=0$$

$$y_{2,3}=\frac{1\pm\sqrt{1-4}}{2}$$

$$y_{2,3}=\frac{1\pm i\sqrt{3}}{2}$$

$$y_2=\frac{1+i\sqrt{3}}{2}$$

$$y_3=\frac{1-i\sqrt{3}}{2}$$

Back in the replacement $x=y\sqrt[3]{\frac{2}{5}}$:

$$x=y\sqrt[3]{\frac{2}{5}}$$

$$x_1=-1\cdot\sqrt[3]{\frac{2}{5}}=-\sqrt[3]{\frac{2}{5}}$$

$$x_2=\frac{1+i\sqrt{3}}{2}\cdot\sqrt[3]{\frac{2}{5}}$$ and $$x_3=\frac{1-i\sqrt{3}}{2}\cdot\sqrt[3]{\frac{2}{5}}$$

Example 4. Solve the equation: $11x^4 - 17 = 0$

Solution:

replacement : $x = y\sqrt[n]{\frac{B}{A}}$ ⟶ $n = 4$, $B = 17$, $A = 11$ $\Rightarrow x = y\sqrt[4]{\frac{17}{11}}$

$$11 \cdot \left(y\sqrt[4]{\frac{17}{11}} \right)^4 - 17 = 0$$

$$11 \cdot y^4 \frac{17}{11} - 17 = 0$$

$$11 \cdot y^4 - 17 = 0 \Rightarrow 17(y^4 - 1) = 0 \Rightarrow y^4 - 1 = 0$$

$$(y^2 - 1)(y^2 + 1) = 0$$

$$(y - 1)(y + 1)(y^2 + 1) = 0$$

$y - 1 = 0$ or $y + 1 = 0$ or $y^2 + 1 = 0$

$y_1 = 1$ or $y_2 = -1$ or $y^2 = -1$

$y_{3,4} = \pm\sqrt{-1} = \pm i$

$y_3 = +i$

$y_4 = -i$

Back in $x = y\sqrt[4]{\frac{17}{11}}$

$x_1 = 1 \cdot \sqrt[4]{\frac{17}{11}} = \sqrt[4]{\frac{17}{11}}$; $x_2 = -1\sqrt[4]{\frac{17}{11}} = -\sqrt[4]{\frac{17}{11}}$

$x_3 = i\sqrt[4]{\frac{17}{11}}$; $x_4 = -i\sqrt[4]{\frac{17}{11}}$

3) Trinomial equations $ax^{2n}+bx^{n}+c=0$

These are the types of equations: $ax^{2n}+bx^{n}+c=0$, where a,b and c are real numbers (different from zero).

These equations can be solved by replacement $x^{n}=t \Rightarrow x^{2n}=t^{2}$.

Example 1. Solve the equation: $x^{6}+7x^{3}-8=0$

Solution:

$$x^{6}+7x^{3}-8=0$$
$$(x^{3})^{2}+7x-8=0 \qquad \text{replacement: } x^{3}=t$$
$$t^{2}+7t-8=0$$
$$t_{1,2}=\frac{-7\pm 9}{2}$$
$$t_{1}=1$$
$$t_{2}=-8$$

Back in the replacement:

$$x^{3}=1$$
$$x^{3}-1=0$$
$$(x-1)(x^{2}+x+1)=0$$
$$x-1=0 \quad \text{v} \quad x^{2}+x+1=0$$
$$x_{1}=1$$
$$x_{2,3}=\frac{-1\pm\sqrt{3}}{2}$$
$$x_{2,3}=\frac{-1\pm i\sqrt{3}}{2}$$

or

$$x^{3}=-8$$
$$x^{3}+8=0$$
$$x^{3}+2^{3}$$
$$(x+2)(x^{2}-2x+4)=0$$
$$x+2=0 \quad \text{v} \quad x^{2}-2x+4$$
$$x_{4}=-2$$
$$x_{5,6}=\frac{2\pm\sqrt{-12}}{2}$$
$$x_{5,6}=\frac{2\pm 2\sqrt{3}i}{2}$$
$$x_{5,6}=1\pm\sqrt{3}i$$

Example 2. Solve the equation: $x^8 + 17x^4 + 16 = 0$

Solution:

$$(x^4)^2 - 17x^4 + 16 = 0 \qquad \text{replacement : } x^4 = t$$

$$x^2 - 17t + 16 = 0$$

$$t_{1,2} = \frac{17 \pm 15}{2}$$

$$t_1 = 16$$

$$t_2 = 1$$

$x^4 = 16$ or $x^4 = 1$

$$x^4 - 16 = 0$$

$$x^4 - 2^4 = 0$$

$$(x^2 - 2^2)(x^2 + 2^2) = 0$$

$$(x-2)(x+2)(x^2+4) = 0$$

$$x - 2 = 0 \ \text{v}\ x + 2 = 0 \ \text{v}\ x^2 + 4 = 0$$

$$x_1 = 2 \quad x_2 = -2 \quad x^2 = -4$$

$$x_{3,4} = \pm\sqrt{-4}$$

$$x_3 = +2i$$

$$x_4 = -2i$$

$$x^4 - 1 = 0$$

$$(x^2 - 1)(x^2 + 1) = 0$$

$$(x-1)(x+1)(x^2+1) = 0$$

$$x - 1 = 0 \ \text{v}\ x + 1 = 0 \ \text{v}\ x^2 + 1 = 0$$

$$x_5 = 1, \quad x_6 = -1, \quad x^2 = -1$$

$$x_{7,8} = \pm\sqrt{-1}$$

$$x_7 = +i$$

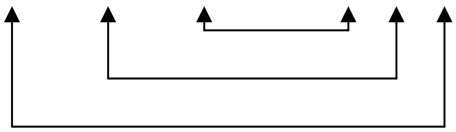
$$x_8 = -i$$

So, solutions are:

$$\{2, -2, 2i, -2i, 1, -1, +i, -i\}$$

4) Symmetrical (reciprocal) equations

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots + cx^2 + bx + a = 0$$



Replacement is: $x + \dfrac{1}{x} = t$

$$\left(x + \frac{1}{x}\right)^2 = t^2$$

$$x^2 + 2 \cdot x \cdot \frac{1}{x} + \frac{1}{x^2} = t^2$$

$$x^2 + 2 + \frac{1}{x^2} = t^2$$

$$x^2 + \frac{1}{x^2} = t^2 - 2 \rightarrow \qquad \textbf{REMEMBER}$$

Sometimes replacement is:

$$\left(x + \frac{1}{x}\right)^3 = t^3$$

$$x^3 + 2x^2 \cdot \frac{1}{x} + 3x + \frac{1}{x^2} + \frac{1}{x^3} = t^3$$

$$x^3 + 3x + 3\frac{1}{x} + \frac{1}{x^3} = t^3$$

$$\boxed{x^3 + 3\left(x + \frac{1}{x}\right) + \frac{1}{x^3} = t^3}$$

$$x^3 + \frac{1}{x^3} = t^3 - 3t \rightarrow \textbf{REMEMBER}$$

etc. ...

Example 1. Solve the equation: $2x^4+3x^3-16x^2+3x+2=0$

Solution:

We share the entire equation with x^2 because it is "intermediate".

$$\frac{2x^4}{x^2}+\frac{3x^3}{x^2}-\frac{-16x^2}{x^2}+\frac{3x}{x^2}+\frac{2}{x^2}=0$$

$$2x^2+3x-16+3\cdot\frac{1}{x}+2\cdot\frac{1}{x^2}=0$$ group members!

$$2\left(x^2+\frac{1}{x}\right)+3\left(x+\frac{1}{x}\right)-16=0$$ replacement: $x+\frac{1}{x}=t$

$$2(t^2-2)+3t-16=0$$
$$2t^2-4+3t-16=0$$
$$2t^2+3t-20=0$$
$$t_{1,2}=\frac{-3\pm\sqrt{9+160}}{4}=\frac{-3\pm 13}{4}$$
$$t_1=-4,\quad t_2=\frac{5}{2}$$

Back in the replacement:

$$x+\frac{1}{x}=-4$$
$$x^2+1=-4x$$
$$x^2+4x+1=0$$
$$x_{1,2}=\frac{-4\pm\sqrt{16-4}}{2}$$
$$x_1=-2+\sqrt{3}$$
$$x_2=-2-\sqrt{3}$$

i

$$x+\frac{1}{x}=\frac{5}{2}$$
$$2x^2+2=5x$$
$$2x^2-5x+2=0$$
$$x_{3,4}=\frac{5\pm\sqrt{25-16}}{4}$$
$$x_3=2$$
$$x_4=\frac{1}{2}$$

Therefore, solutions are 2 and $\frac{1}{2}$; $-2+\sqrt{3}$ and $-2-\sqrt{3}$. Are they reciprocal?

It is obviously for 2 and $\frac{1}{2}$ and what is with $-2+\sqrt{3}$ and $-2-\sqrt{3}$?

$$\frac{-2+\sqrt{3}}{1}=\frac{-2+\sqrt{3}}{1}\cdot\frac{-2-\sqrt{3}}{-2-\sqrt{3}}=\frac{(-2)^2-\sqrt{3}}{-2-\sqrt{3}}=\frac{1}{-2-\sqrt{3}}$$

Now (after rationalization),we see that it is also reciprocal.

Example 2: Solve the equation: $12x^5+16x^4-37x^3-37x^2+16x+12=0$

Solution:

This is the fifth degree equation, and one solution is obviously $x=-1$.Whole equation we will share with $(x+1)$.

$$(12x^5+16x^4-37x^3-37x^2+16x+12):(x+1)=12x^4+4x^3-41x^2+4x+12$$

View polynomials sharing!!!

$$12x^4+4x^3-41x^2+4x+12=0 / : x^2$$

$$12x^2+4x-41+4\cdot\frac{1}{x}+12\cdot\frac{1}{x^2}=0$$

$$12\left(x^2+\frac{1}{x^2}\right)+4\left(x+\frac{1}{x}\right)-41=0 \quad \text{replacement} : x+\frac{1}{x}=t \Rightarrow x^2+\frac{1}{x^2}=t^2-2$$

$$12(t^2-2)+4t-41=0$$

$$12t^2-24+4t-41=0$$

$$12t^2+4t-65=0$$

$$t_{1,2}=\frac{-4\pm 56}{24}$$

$$t_1=\frac{13}{6}$$

$$t_2=-\frac{5}{2}$$

Back in the replacement:

$$x+\frac{1}{x}=\frac{13}{6}$$

$$6x^2-13x+6=0$$

$$x_{1,2}=\frac{13\pm 5}{12}$$

$$x_1=\frac{18}{12}=\frac{3}{2}$$

$$x_2=\frac{8}{12}=\frac{2}{3}$$

and

$$x+\frac{1}{x}=-\frac{5}{2}$$

$$2x^2+5x+2=0$$

$$x_{3,4}=\frac{-5\pm 3}{4}$$

$$x_3=-\frac{1}{2}$$

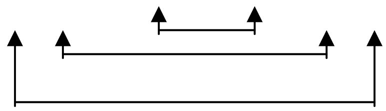
$$x_4=-2$$

So: $\left\{\frac{3}{2},\frac{2}{3},-\frac{1}{2},-2,-1\right\}$ are solutions!

Example 2: Solve the equation: $x^5 - 7x^4 + 16x^3 - 16x^2 + 7x - 1 = 0$

Solution:

$$x^5 - 7x^4 + 16x^3 - 16x^2 + 7x - 1 = 0$$



One solution is $x_1 = 1$. The whole equation we share with $(x-1)$

$$(x^5 - 7x^4 + 16x^3 - 16x^2 + 7x - 1):(x-1) = x^4 - 6x^3 + 10x^2 - 6x + 1$$

$x^4 - 6x^3 + 10x^2 - 6x + 1 = 0$ it is symmetrical $/ : x^2$

$$x^4 - 6x^3 + 10x^2 - 6x + 1 = 0$$
$$x^2 - 6x + 10 - 6 \cdot \frac{1}{x} + \frac{1}{x^2} = 0$$

Solutions are: $x_1 = 1, x_2 = 1, x_3 = 2 + \sqrt{3}, x_4 = 2 - \sqrt{3}$ and $x_5 = 1$